On Galilean invariance and nonlinearity in electrodynamics and quantum mechanics

Gerald A. Goldin

Departments of Mathematics and Physics, Rutgers University, Busch Campus, Piscataway, New Jersey 08854

and

Vladimir Shtelen

Department of Mathematics, Rutgers University, Busch Campus, Piscataway, New Jersey 08854

3 May 2000

Abstract

Recent experimental results on slow light heighten interest in nonlinear Maxwell theories. We obtain Galilei covariant equations for electromagnetism by allowing special nonlinearities in the constitutive equations only, keeping Maxwell's equations unchanged. Combining these with linear or nonlinear Schrödinger equations, e.g. as proposed by Doebner and Goldin, yields a Galilean quantum electrodynamics.

MSC: 78A25, 78A97, 81B05, 81G10.

Key words: nonlinear electrodynamics, Schrödinger equation, Galilean symmetry.

1 Introduction

In this letter we propose nonlinear constitutive equations restricting the symmetry of Maxwell's equations to Galilean symmetry. We also show that these constitutive equations arise as the formal nonrelativistic limit (taking the speed of light $c \to \infty$) of the nonlinear relativistic theory: no additional hypotheses about relative field strengths are needed. Admitting such nonlinear constitutive equations leads to possibilities that are entirely new. Maxwell's equations in our approach stay unchanged; it is the choice of the constitutive equations alone that makes the difference between relativistic and nonrelativistic theories. We stress that the resulting Galilean electrodynamics is essentially nonlinear, although gauge invariant; linear constitutive equations are indeed incompatible with Maxwell's equations and Galilean covariance. Further, we extend the Galilean symmetry to the minimally-coupled Schrödinger-Maxwell theory, and to the coupled systems of Maxwell and nonlinear Schrödinger equations as proposed by Doebner and Goldin. Thus one has a consistent and fully Galilean covariant (but nonlinear) quantum electrodynamics.

There is a commonly held "folk belief" among physicists that Maxwell's equations, being Lorentz invariant, are not consistent with Galilean symmetry. However it is known (but not widely appreciated) that Galilean covariance, like Lorentz covariance, is a property of all four of Maxwell's equations for media in classical electrodynamics. More than 25 years ago, Le Bellac and Lévy-Leblond emphasized this in [1], noting that the clash between Maxwell's equations and Galilean relativity occurs only in the constitutive equations. Keeping the standard constitutive equations for the vacuum, and introducing some additional conditions for the electromagnetic fields, these authors derived two distinct Galilean limits of Maxwell's equations: one in which Faraday's law is lost (called the electric limit); and another in which the displacement current is zero, that violates the continuity equation for charge and current densities (called the magnetic limit). Brown and Holland, in a recent discussion of these results, observed as follows: "It is noted that no fully Galilean-covariant theory of a coupled Schrödinger-Maxwell system (where the density and current of the Schrödinger field act as source of the nonrelativistic Maxwell field) is possible." [2]

Dyson, in discussing an unpublished 1948 "proof" of Maxwell's equations by Feynman, remarks, "The proof begins with assumptions invariant under Galilean transformations and ends with equations invariant under Lorentz transformations. How could this have happened?" [3]. Dyson's paper immediately provoked a heated discussion in the literature; see, for example [4, 5, 6]. In particular, Vaidya and Farina [6] ask in the title of their paper, "Can Galilean mechanics and full Maxwell equations coexist peacefully?" and answer forcefully, "No, they cannot."

The fact that Maxwell's equations for media are both Lorentz and Galilei invariant sheds light on the mystery behind Feynman's insight. The conclusions of Brown and Holland, and Vaidya and Farina, depend on the implicit assumption of linear constitutive equations. In this letter we take a completely different point of view. We show how the choice between a Lorentz or Galilei invariant theory can be made by introducing a special class of nonlinearities in the constitutive equations, *without modifying any of Maxwell's equations (or the continuity equation).* Unlike Le Bellac and Lévy-Leblond we are not forced by constitutive equations to choose between a privileged frame of reference (the ether) and Einsteinian relativity.

Our interest in nonlinearity and Maxwell's equations is heightened by recent experimental results in quantum optics, including optical squeezing and quantum nondemolition measurements, that suggest the fundamental importance of nonlinear constitutive equations in media. In particular, we are motivated by recent experimental results demonstrating extremely slow light speed (as slow as 17 m/sec) in laser-dressed ultra-cold atomic media [7] to look again at the question of Galilean-invariant electrodynamics in the context of nonlinear, coupled Schrödinger-Maxwell theories. The result is the framework for Galilean quantum electrodynamics described here. We would like to raise the question whether the constitutive equations in our Galilean class could be a good mathematical model for some real media, in some range of velocities and field strengths.

2 Symmetry of Maxwell's equations for media

A systematic investigation of symmetry of Maxwell's equations for media was reported in Refs. [8, 9], where it was shown that these equations are covariant under the inhomogeneous group of general linear transformations $GL(4, \mathbf{R})$, that includes both Lorentz and Galilei transformations. General constitutive equations restricting this symmetry to Poincaré symmetry were found.

It is important for further analysis to use a system of units such as the SI system, that avoids incorporating c into the definition of the fundamental fields. Without such a choice it is not possible to see how nonrelativistic theory arises from a relativistic theory in the limit $c \to \infty$. Maxwell's equations for media written in SI units have the form [10, 11, 12]:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0; \\ \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \\ \nabla \cdot \mathbf{D} &= \rho, \end{aligned} \tag{2.1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction, and $\mathbf{H}$ is the magnetic field; ρ and $\mathbf{j}$ are charge and current densities. The physically detectable fields are $\mathbf{E}$ and $\mathbf{B}$, via the Lorentz force on a charged particle $\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$.

The fields **H** and **D** may be regarded as constructs used to describe (via the constitutive equations) how the directly observable fields are produced by charges and currents.

As is well known, Maxwell's equations are Lorentz covariant; in particular, they are covariant under the space-time transformations [10, 11, 12]

$$x'_{\|} = \gamma(\mathbf{x} - \mathbf{v}t)_{\|}, \quad x'_{\perp} = x_{\perp}, \quad t' = \gamma(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}), \tag{2.2}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}};$$

with the corresponding field transformations

$$\begin{aligned} B'_{\|} &= B_{\|}, \quad B'_{\perp} = \gamma(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E})_{\perp}, \\ E'_{\|} &= E_{\|}, \ E'_{\perp} = \gamma(\mathbf{E} + \mathbf{v}\times\mathbf{B})_{\perp}, \\ H'_{\|} &= H_{\|}, \ H'_{\perp} = \gamma(\mathbf{H} - \mathbf{v}\times\mathbf{D})_{\perp}, \\ D'_{\|} &= D_{\|}, \ D'_{\perp} = \gamma(\mathbf{D} + \frac{1}{c^2}\mathbf{v}\times\mathbf{H})_{\perp}; \end{aligned} \tag{2.3}$$

and the transformations for current and charge densities:

$$j'_{\|} = \gamma(\mathbf{j} - \mathbf{v}\rho)_{\|}, \quad j'_{\perp} = j_{\perp}, \quad \rho' = \gamma(\rho - \frac{\mathbf{v}\cdot\mathbf{j}}{c^2}). \tag{2.4}$$

It is appropriate to list here the Lorentz field invariants:

$$\begin{aligned} I_1 &= \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2, \quad I_2 = \mathbf{B}\cdot\mathbf{E}; \\ I_3 &= \mathbf{D}^2 - \frac{1}{c^2}\mathbf{H^2}, \quad I_4 = \mathbf{H}\cdot\mathbf{D}; \\ I_5 &= \mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D}, \quad I_6 = \mathbf{B}\cdot\mathbf{D} + \frac{1}{c^2}\mathbf{E}\cdot\mathbf{H}. \end{aligned} \tag{2.5}$$

One can also verify that Maxwell's equations (2.1) are covariant with respect to the Galilean transformations

$$t' = t, \quad \mathbf{x}' = \mathbf{x} - \mathbf{vt};$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \quad \mathbf{B}' = \mathbf{B},$$

$$\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}, \quad \mathbf{D}' = \mathbf{D}; \tag{2.6}$$

$$\mathbf{j}' = \mathbf{j} - \rho\mathbf{v}, \quad \rho' = \rho,$$

which arise as the $c \to \infty$ limit of the Lorentz transformations (2.2) - (2.4). Both Galilean and Lorentz symmetries belong to the class of general linear transformations $GL(4, \mathbf{R})$ admitted by Maxwell's equations (2.1). This, in turn, is a consequence of the incompleteness of the system (2.1): there are 8 equations for 12 unknown functions. The system must be completed by the constitutive equations, which are functional relations between vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$.

A particular choice of the constitutive equations can reduce the symmetry of the system (2.1) to Lorentz or to Galilei. General constitutive equations that restrict the $GL(4, \mathbf{R})$ symmetry of Maxwell's equations to Lorentz (Poincaré) are reported in Refs. [8, 9]. In SI units, they have the form

$$\mathbf{D} = M\mathbf{B} + \frac{1}{c^2}N\mathbf{E}, \quad \mathbf{H} = N\mathbf{B} - M\mathbf{E}, \tag{2.7}$$

where M and N are arbitrary scalar functions of the Lorentz invariants I_1, I_2 given in (2.5). Equivalently we may write

$$\mathbf{B} = R\mathbf{D} + \frac{1}{c^2}Q\mathbf{H}, \quad \mathbf{E} = Q\mathbf{D} - R\mathbf{H}, \tag{2.8}$$

where Q and R are arbitrary functions of the Lorentz invariants I_3, I_4 from (2.5).

Let us note that some well-known nonlinear theories like Born-Infeld electrodynamics, or Euler-Kockel electrodynamics that takes into account quantum-mechanical nonlinear effects (see, for example, [10] and references therein), correspond to particular choices of M and N in (2.7).

The constitutive equations that reduce the $GL(4, \mathbf{R})$ symmetry group of (2.1) to the Galilei group are

$$\mathbf{D} = \hat{M}\mathbf{B}, \quad \mathbf{H} = \hat{N}\mathbf{B} - \hat{M}\mathbf{E}, \tag{2.9}$$

or, equivalently

$$\mathbf{B} = \hat{R}\mathbf{D}, \quad \mathbf{H} = \hat{Q}\mathbf{D} - \hat{R}\mathbf{H}, \tag{2.10}$$

where $\hat{M}$ and $\hat{N}$, $\hat{Q}$ and $\hat{R}$ are arbitrary functions of Galilean invariants. To demonstrate this, we first use (2.6) to see that the general constitutive equations $\mathbf{D} = \mathbf{f}(\mathbf{B}, \mathbf{E})$, $\mathbf{H} = \mathbf{g}(\mathbf{B}, \mathbf{E})$ must take the form $\mathbf{D} = \hat{M}\mathbf{B}, \quad \mathbf{H} = \hat{N}\mathbf{B} + \hat{N}_1\mathbf{E}$, where $\hat{M}, \hat{N}, \hat{N}_1$ are some scalar functions of Galilean invariants. Then substitution of the latter equation into (2.6) results in the condition $\hat{N}_1 = -\hat{M}$. Finally, one shows that the Galilean field invariants are:

$$\begin{gathered} \hat{I}_1 = \mathbf{B}^2, \quad \hat{I}_2 = \mathbf{B} \cdot \mathbf{E}; \\ \hat{I}_3 = \mathbf{D}^2, \quad \hat{I}_4 = \mathbf{H} \cdot \mathbf{D}; \\ \hat{I}_5 = \mathbf{B} \cdot \mathbf{H} - \mathbf{E} \cdot \mathbf{D}, \quad \hat{I}_6 = \mathbf{B} \cdot \mathbf{D}. \end{gathered} \tag{2.11}$$

One can, if one wishes, restrict oneself to constitutive equations in explicit (i.e., non-implicit) form, so that $\hat{M}$ and $\hat{N}$ depend on $\hat{I}_1, \hat{I}_2$, and $\hat{Q}$ and $\hat{R}$ depend on $\hat{I}_3, \hat{I}_4$.

Note that the constitutive equations (2.9), (2.10) and the Galilean field invariants (2.11) are respectively the formal limits as $c \to \infty$ of the Lorentz invariant relations (2.7), (2.8), and (2.5).

One obtains the standard Maxwell equations for the vacuum (where $c^{-2} = \epsilon_0\mu_0$) by choosing $N = 1/\mu_0$, $M = 0$ in (2.7). But in the case of Galilean constitutive equations, letting M be a constant is not compatible with a nonzero charge density ρ. Therefore, we conclude that a consistent Galilean electrodynamics is *essentially nonlinear.*

It is worth remarking for the case of relativistic constitutive equations (2.7), that letting $M = \lambda =$ constant, so that

$$\mathbf{D} = \lambda\mathbf{B} + \frac{1}{c^2}N\mathbf{E}, \quad \mathbf{H} = N\mathbf{B} - \lambda\mathbf{E},$$

results in equations for the observable fields $\mathbf{B}$ and $\mathbf{E}$ that are independent of the magnitude of λ. In other words, constant terms λ in the above constitutive equations have no effect on observable fields, and therefore, without loss of generality one can set $\lambda = 0$. In general, adding a constant to M in (2.7) has no effect on the observable fields $\mathbf{E}$ and $\mathbf{B}$. This is also true for $\hat{M}$ in (2.9), which is why nontrivial Galilean constitutive equations are essentially nonlinear.

The (linear) Galilean electrodynamics discussed in the literature is associated with the pre-Maxwell electrodynamics, with no displacement current [1, 2, 10, 12], it is also called the "magnetic limit" of Maxwell's equations, and is the only one of the two limits discussed in [1] that is compatible with Schrödinger quantum mechanics. These equations in SI units have the form:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} &= 0; \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j}, \\ \nabla \cdot \mathbf{E} &= {\epsilon_0}^{-1} \rho. \end{aligned} \tag{2.12}$$

Let us note, that the approach used in [1] to derive (2.12) from (2.1) actually embodies contradictory assumptions, moving back and forth between the two symmetries. First, the constitutive equations for the vacuum $\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu_0 \mathbf{H}$ are imposed, immediately breaking the Galilean symmetry of Maxwell's equations (2.1). Then additional, non-invariant conditions about relative field strengths are assumed to justify the limit $1/c^2 \to 0$ in Maxwell's equations, and remove the displacement current. At the same time both ϵ_0 and μ_0 are kept non-zero, even though their product $1/c^2$ has been taken to zero. Since there is no displacement current in (2.12), the continuity equation does not hold. The Galilean transformations for $\mathbf{E}$ and $\mathbf{B}$ in (2.12) are as in (2.6), while the corresponding transformations for ρ and $\mathbf{j}$ are

$$\mathbf{j}' = \mathbf{j}, \quad \rho' = \rho - \epsilon_0 \mu_0 \mathbf{v} \cdot \mathbf{j}. \tag{2.13}$$

In order to explore the consistency of the Galilean electrodynamics (2.1), (2.9), we have investigated two situations: first, the case $\hat{M} = 0$, $\hat{N} = \mathbf{E}\cdot\mathbf{B}$ (which requires $\rho = 0$); secondly, $\hat{M} = \mathbf{E}\cdot\mathbf{B}$, $\hat{N} = 1/\mu_0$. We were able to obtain some particular solutions, and this is a subject of our ongoing research. Let us note here that using the Galilei invariance of the system, one can construct (new) traveling wave solutions from (old) solutions by means of the following formulas:

$$\begin{gathered}\mathbf{E}_{new}(x) = \mathbf{E}_{old}(x') - \mathbf{v}\times\mathbf{B}_{old}(x'), \quad \mathbf{B}_{new}(x) = \mathbf{B}_{old}(x');\\ \mathbf{H}_{new}(x) = \mathbf{H}_{old}(x') + \mathbf{v}\times\mathbf{D}_{old}(x'), \quad \mathbf{D}_{new}(x) = \mathbf{D}_{old}(x');\\ \mathbf{j}_{new}(x) = \mathbf{j}_{old}(x') + \mathbf{v}\rho_{old}(x'), \quad \rho_{new}(x) = \rho_{old}(x'),\end{gathered} \tag{2.14}$$

where $x' = (\mathbf{x}', t') = (\mathbf{x} - \mathbf{v}t, t)$.

3 A framework for a Galilei covariant quantum electrodynamics

As is well known, any solution for vectors $\mathbf{E}$ and $\mathbf{B}$ of Maxwell's equations (2.1), can be represented in terms of potentials $(\Phi,\ \mathbf{A})$

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A},\\ \mathbf{E} &= -\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi.\end{aligned} \tag{3.1}$$

for some Φ and $\mathbf{A}$. The choice of Φ and $\mathbf{A}$ is not unique: new potentials $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$, $\Phi' = \Phi - \partial\Lambda/\partial t$, containing an arbitrary function Λ, result in the same $\mathbf{E}$ and $\mathbf{B}$ (gauge invariance). The standard procedure of introducing electromagnetic interaction in quantum mechanics is minimal coupling, which is consistent with Galilean covariance of the free field equations. One obtains the coupled Schrödinger-Maxwell equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - e\mathbf{A})^2\psi + e\Phi\psi, \tag{3.2}$$

where the electromagnetic fields $\mathbf{E}, \mathbf{B}$ are obtained from $\Phi, \mathbf{A}$ via (3.1), and are governed by (2.1). Gauge invariant current and charge densities entering (2.1) are given by

$$\rho = \bar{\psi}\psi, \quad \mathbf{J}^{\mathbf{gi}} = \frac{\hbar}{2im}[\bar{\psi}\nabla\psi - (\nabla\bar{\psi})\psi] - \frac{e}{m}\rho\mathbf{A}. \tag{3.3}$$

Equation (3.2) is Galilean invariant, with corresponding transformations given in (2.6), and the transformations for vector-potential $(\Phi, \mathbf{A})$ given by

$$\mathbf{A}' = \mathbf{A}, \quad \Phi' = \Phi - \mathbf{v} \cdot \mathbf{A}. \tag{3.4}$$

The resulting fully Galilean covariant quantum electrodynamics is embodied in equations (2.1), (2.9) (with some concrete choices of $\hat{M}$ and $\hat{N}$), (3.1), (3.2), (3.3).

It is natural to ask about the coupling of nonlinear Maxwell theory with nonlinear Galilean invariant Schrödinger mechanics. A possible framework for such a quantum mechanics is given by the family of nonlinear equations proposed by Doebner and Goldin [13]. Nonlinear terms of the form

$$\frac{i\hbar D}{2}\frac{\triangle\rho}{\rho}\psi + \hbar D'[c_1\frac{\nabla \cdot \hat{\mathbf{j}}}{\rho} + c_2\frac{\triangle\rho}{\rho} + c_3\frac{\hat{\mathbf{j}}^2}{\rho^2} + c_4\frac{\hat{\mathbf{j}} \cdot \nabla\rho}{\rho^2} + c_5\frac{(\nabla\rho)^2}{\rho^2}]\psi \tag{3.5}$$

are added to the right side of Eq.(3.2), where $\hat{\mathbf{j}} = (1/2i)[\bar{\psi}\nabla\psi - (\nabla\bar{\psi})\psi]$, and D, D' are diffusion coefficients. The Galilean-invariant subfamily of quantum theories is defined by $c_1 + c_4 = c_3 = 0$. We are proposing consideration of the coupled system of Doebner-Goldin equations and nonlinear Galilean invariant Maxwell equations, as a consistent Galilean quantum electrodynamics. In this system, the quantities $\rho, \mathbf{E}$, $\mathbf{B}$, and

$$\mathbf{J}^{\mathbf{gi}} = \frac{\hbar}{2im}[\bar{\psi}\nabla\psi - (\nabla\bar{\psi})\psi] - D\nabla\rho - \frac{e}{m}\rho\mathbf{A}. \tag{3.6}$$

are gauge invariant. Eq.(3.6) gives the current to be entered in Maxwell's equations. Doebner and Goldin further generalize the notion of gauge transformation to allow nonlinear gauge transformations, writing more general formulas for the gauge invariants $\rho, \mathbf{J}^{\mathbf{gi}}, \mathbf{E}$, and **B** [14]. Recently Galilean invariant Doebner-Goldin equations were applied to describe the dynamics of matrix D-branes [16].

In conclusion, let us remark that one can extend our approach further to include in the theory Galilei invariant spinor equations (see, for example, [8]). Another possible generalization of the quantum electrodynamics described here is to non-Abelian Yang-Mills fields.

References

[1] M. Le Bellac and J.-M. Levy-Leblond, Galilean electromagnetism, Nuovo Cim. **14B** (1973) 217.

[2] H. R. Brown, P. R. Holland, The Galilean covariance of quantum mechanics in the case of external fields, Am. J. Phys. **67** (1999) 204.

[3] F. J. Dyson, Feynman's proof of the Maxwell equations, Am. J. Phys **58** (1990) 209.

[4] N. Dombey, Comment on "Feynman's proof of the Maxwell equations", by F.J. Dyson, Am. J. Phys **59** (1991) 85.

[5] R. J. Hughes, On Feynman's proof of the Maxwell equations, Am. J. Phys **60** (1991) 301.

[6] A. Vaidya and C. Farina, Can Galilean mechanics and full Maxwell equations coexist peacefully?, Phys. Lett. A **153** (1991) 265.

[7] J. Marangos, Slow light in cool atoms, Nature **397** (18 February 1999), 559.

[8] W. I. Fushchich, V. M. Shtelen, and N. I. Serov, *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics,* Kluwer Acad. Publ., Dordrecht (1993).

[9] W. Fushchich and I. Tsifra, On symmetry of nonlinear equations of electrodynamics, Teor. Mat. Fizika **64** (1985) 41.

[10] J. D. Jackson, *Classical Electrodynamics,* third edition, J. Wiley & Sons, Inc., New York (1999).

[11] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles,* Dover Publ., New York (1980).

[12] J. Vanderlinde, *Classical Electromagnetic Theory,* Wiley, New York (1993).

[13] H.-D. Doebner and G. A. Goldin, Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations, Phys. Rev. A **54** (1996) 3764.

[14] G. A. Goldin, Gauge transformations for a family of nonlinear Schrödinger equations, J. Nonl. Math. Phys. **4** (1997) 6.

[15] H.-D. Doebner, G. A. Goldin, and P. Nattermann, Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations, J. Math. Phys. **40** (1999) 49.

[16] N. E. Mavromatos and R. S. Szabo, Nonlinear Schrödinger dynamics of matrix D-branes, Preprint (September 1999), OUTP-99-47P, CERN-TH/99-256, NBI-HE-99-34, hep-th/9909129.